$$\Rightarrow 2\sum_{1\le k\le p}\left(\frac{\pi}{2}\right)+2\left(\frac{\pi}{4}\right)-\frac{\pi}{2}$$

(OR)

$$S=\pi p$$

(Answer)

Shivam Sharma

Third solution. Corrected setting:
Let p be an integer and a positive real number. Prove that

$$\sum_{n=-\infty}^{\infty}\arctan\left(\frac{a^p-a^{-p}}{a^n+a^{-n}}\right)=\begin{cases}\pi\text{if } a>1\\ 0 \text{ if } a=1\\ -n\pi \text{ if } a\in(0,1)\end{cases}.$$

Obvious that for $a=1$ the sum equal to zero.
Let $a>1$ and let $\alpha_n:=\arctan a^n, n\in\mathbb{Z}$. Then $a^n=\tan\alpha_n, n\in\mathbb{Z}$ and

$$\arctan\left(\frac{a^p-a^{-p}}{a^n+a^{-n}}\right)=\arctan\left(\frac{a^n\left(a^p-a^{-p}\right)}{a^n\left(a^n+a^{-n}\right)}\right)=\arctan\left(\frac{a^{n+p}-a^{n-p}}{1+a^{2n}}\right)=$$

$$=\arctan\left(\frac{a^{n+p}-a^{n-p}}{1+a^{n+p}\cdot a^{n-p}}\right)=\arctan\left(\tan\left(\alpha_{n+p}-\alpha_{n-p}\right)\right)=\alpha_{n+p}-\alpha_{n-p}$$

(because $\alpha_n\in(0,\infty), n\in\mathbb{Z}\implies\alpha_{n+p}-\alpha_{n-p}\in(-\pi/2,\pi/2), n,p\in\mathbb{Z}$).
Let $\beta_n:=\alpha_{n+p}+\alpha_{n+p-1}+\ldots+\alpha_{n-p+1}, n\in\mathbb{Z}$. Since
$\alpha_{n+p}-\alpha_{n-p}=\beta_n-\beta_{n-1}$ then

$$S=\lim_{n\to\infty}\sum_{k=-n}^{n}\arctan\left(\frac{a^p-a^{-p}}{a^k+a^{-k}}\right)=\lim_{n\to\infty}\sum_{k=-n}^{n}\left(\alpha_{k+p}-\alpha_{k-p}\right)=$$

$$=\lim_{n\to\infty}\sum_{k=-n}^{n}\left(\beta_k-\beta_{k-1}\right)=$$

$$=\lim_{n\to\infty}\left(\beta_n-\beta_{-n-1}\right)=$$

$$=\lim_{n\to\infty}\beta_n-\lim_{n\to\infty}\beta_{-(n+1)}=\lim_{n\to\infty}\left(\alpha_{n+p}+\alpha_{n+p-1}+\ldots+\alpha_{n-p+1}\right)-$$

$$-\lim_{n\to\infty}(\alpha_{-n-1+p}+\alpha_{-n+p-2}+...+\alpha_{-n-p}) = 2p\cdot\frac{\pi}{2}-2p\cdot 0=\pi p$$

because $\lim_{n\to\infty}\alpha_{n+k}=\lim_{n\to\infty}\arctan a^{n+k}=\frac{\pi}{2}$ and $\lim_{n\to\infty}\arctan a^{-n+k}=0$ for any integer k.

If $a\in(0,1)$ then $\frac{1}{a}>1$ and

$$\sum_{n=-\infty}^{\infty}\arctan\left(\frac{a^p-a^{-p}}{a^n+a^{-n}}\right)=$$

$$=-\sum_{n=-\infty}^{\infty}\arctan\left(\frac{(1/a)^p-(1/a)^{-p}}{(1/a)^n+(1/a)^{-n}}\right)=-\pi p.$$

Arkady Alt

Fourth solution. The result is partially correct. It should be

$$\pi p\frac{a-1}{|a-1|}\text{ if } a\neq 1,\quad 0 \text{ if } a=1$$

If $a=1$ we have

$$\arctan\frac{a^p-a^{-p}}{a^n+a^{-n}}=\arctan\frac{1-1}{a^n+a^{-n}}\equiv 0$$

Let $a>1$. We have

$$\sum_{n=-\infty}^{\infty}\arctan\left(\frac{a^p-a^{-p}}{a^n+a^{-n}}\right)=\arctan\frac{a^p-a^{-p}}{2}+2\sum_{n=1}^{\infty}\arctan\left(\frac{a^p-a^{-p}}{a^n+a^{-n}}\right)=$$

$$=\arctan\frac{a^p-a^{-p}}{2}+2\sum_{n=1}^{\infty}\arctan\left(\frac{a^{n+p}-a^{n-p}}{1+a^{2n}}\right)=$$

$$=\arctan\frac{a^p-a^{-p}}{2}+2\lim_{N\to\infty}\sum_{n=1}^{N}\left(\arctan a^{n+p}-\arctan a^{n-p}\right)=$$

$$=\arctan\frac{a^p-a^{-p}}{2}+2\lim_{N\to\infty}\left[\sum_{n=p+1}^{N+p}\arctan a^k-\sum_{n=1-p}^{N-p}\arctan a^k\right]=$$